Measurable selectors and set-valued Pettis integral in non-separable Banach spaces

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The papers

- **B. Cascales and J. Rodríguez**, *The Birkhoff integral and the property of Bourgain*, Math. Ann. **331 (2005)**, no. 2, 259–279. MR 2115456
- **B. Cascales and J. Rodríguez**, *Birkhoff integral for multi-valued functions*, J. Math. Anal. Appl. **297 (2004)**, no. 2, 540–560, Special issue dedicated to John Horváth. MR 2088679 (2005f:26021)
- **B. Cascales, V. Kadets, and J. Rodríguez**, *The Pettis integral for multi-valued functions via single-valued ones*, J. Math. Anal. Appl. **332 (2007)**, no. 1, 1–10.
- **B. Cascales, V. Kadets, and J. Rodríguez**, *Measurable selectors and set-valued Pettis integral in non-separable Banach spaces*, Submitted **(2007)**.

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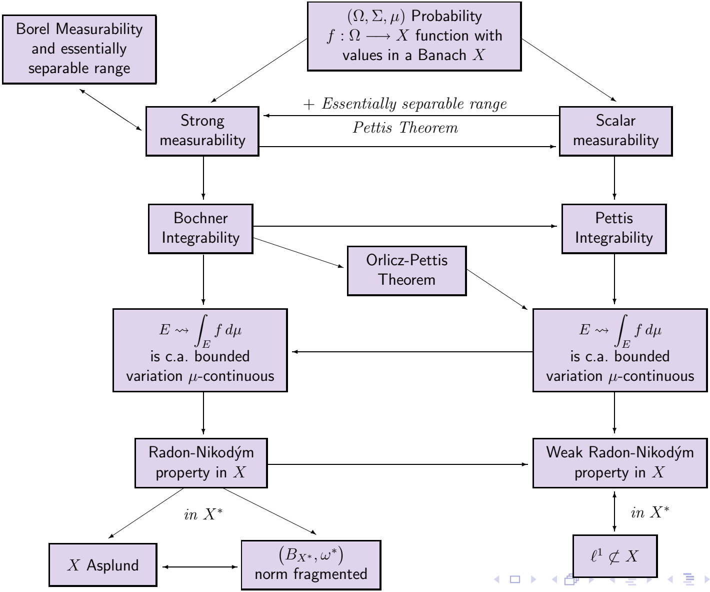
- $L^1(\mu)$: μ-integrable real functions.

Integral de Bochner e Integral de Pettis



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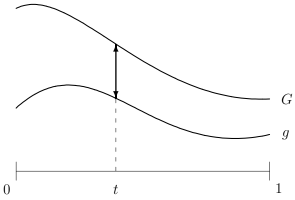
$F : \Omega \longrightarrow cwk(X)$ –convex w-compact



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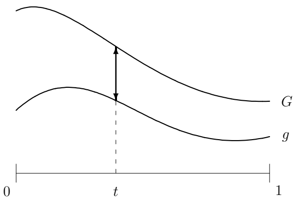

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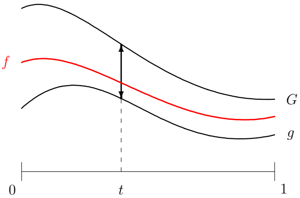

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$$\int F\,d\mu = \left\{ \int f\,d\mu : f \text{ integra. sel.} F \right\}.$$

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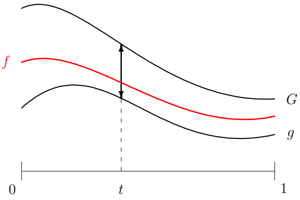

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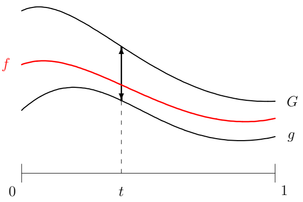

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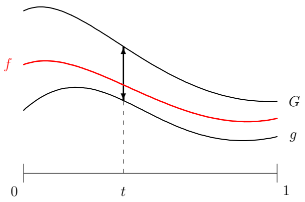

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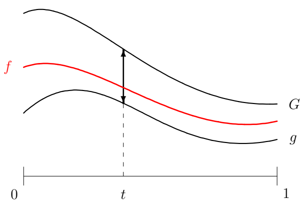

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4. Debreu Nobel prize in 1983; Aumann Nobel prize in 2005.

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An integration theory of vector-valued functions (Bochner, Pettis, McShane, etc.) and

- a good embedding result $j : cwk(X) \to Y$;
- a good selection theorem;
- or all the above.

Hausdorff distance and Rådström embedding[Bar93, CV77, KT84, Råd52]

Definition

Take $C, D \subset X$ bounded sets. The Hausdorff distance between C and D is

$$h(C,D) := \inf\{\eta > 0 : \ C \subset D + \eta B_X, \ D \subset C + \eta B_X\}.$$

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5. X is separable, if and only if, $(ck(X), h)$ is separable.

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6. If $C_n \xrightarrow{h} C$ in $\mathscr{C}$ then

$$C := \{x \in X : \text{existe } x_n \in C_n \text{ con } x = \lim_n x_n\}$$

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Theorem, Rådström embedding [Råd52]

The map $j : cwk(X) \longrightarrow \ell_\infty(B_{X^*})$ given by por $j(C)(x^*) = \delta^*(x^*, C)$ satisfies the following properties:

(i) $j(C+D) = j(C) + j(D)$ for each $C, D \in cwk(X)$;

(ii) $j(\lambda C) = \lambda j(C)$ for each $\lambda \geq 0$ and $C \in cwk(X)$;

(iii) $h(C,D) = \|j(C) - j(D)\|_\infty$ for each $C, D \in cwk(X)$;

(iv) $j(cwk(X))$ is closed in $\ell_\infty(B_{X^*})$.

Kuratowski y Ryll-Nardzewski selection theorem

Definition

$F:\Omega \longrightarrow \mathscr{C}(X)$ is said to be (Effros) measurable

$$\{t\in\Omega:\ F(t)\cap F\neq\emptyset\}\in\Sigma \quad \text{for every closed subset } F\subset X.$$

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Theorem, Kuratowski and Ryll-Nardzewski [KRN65]

If X is separable, then every measurable multi-función $F : \Omega \longrightarrow \mathscr{C}(X)$ has a measurable selector.

Debreu integral

Definition

$F:\Omega \longrightarrow ck(X)$ is said to be Debreu integrable if $j\circ F:\Omega \longrightarrow \ell_\infty(B_{X^*})$ is Bochner integrable.

Remark

The above conditions imply that there is a unique $C \in ck(X)$ satisfying $j(C) = (\text{Bochner})\int_\Omega j\circ F\, d\mu$. By definition:

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Definition

Let $F:\Omega \longrightarrow cwk(X)$ be a multi-function.

- If $x^* \in X^*$ we write $\delta^*(x^*,F)$ to denote the real function defined on Ω by
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- If $F : \Omega \longrightarrow ck(X)$ es Debreu integrable, then F has Bochner integrable selectors.

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Remark

If $F:\Omega \longrightarrow ck(X)$ Debreu integrable we always can assume X separable.

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Theorem

Si $F : \Omega \longrightarrow ck(X)$ is Debreu integrable then

$$(D)\int_{\Omega} F d\mu = \{\int_{\Omega} f d\mu : f \text{ measurable selector for } F\}.$$

Integral de Pettis

Definición

Let X be a separable Banach space. A multi-function $F : \Omega \to cwk(X)$ is said to be *Pettis integrable* if

- $\delta^*(x^*, F)$ is integrable for each $x^* \in X^*$;
- for each $A \in \Sigma$, there is $\int_A F \, d\mu \in cwk(X)$ such that

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Remark – $j \circ F : \Omega \longrightarrow \ell_\infty(B_{X^*})$

If X is separable, $F : \Omega \longrightarrow cwk(X)$ is Pettis integrable iff

(i) $\langle e_{x^*}, j \circ F \rangle \in \mathscr{L}^1(\mu)$ for every $x^* \in X^*$;

(ii) for each $A \in \Sigma$, there is $(P)\int_A F \, d\mu \in cwk(X)$ such that

$$\langle e_{x^*}, j\left((P)\int_A F \, d\mu\right) \rangle = \int_A \langle e_{x^*}, j \circ F \rangle \, d\mu \quad \text{para todo } x^* \in X^*.$$

Set-valued Pettis integration

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The Milestone result, [EAH00, Zia97, Zia00] and [CV77, Chapter V, §4]

If X is a separable Banach space and $F : \Omega \to cwk(X)$ a multi-function TFAE:

(i) F is Pettis integrable.

(ii) The family $W_F = \{\delta^*(x^*, F) : x^* \in B_{X^*}\}$ is uniformly integrable.

(iii) The family W_F is made up of measurable functions and any scalarly measurable selector of F is Pettis integrable.

In this case, for each $A \in \Sigma$ the integral $\int_A F \, d\mu$ coincides with the set of integrals over A of all Pettis integrable selectors of F.

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In this case, for each $A \in \Sigma$ the integral $\int_A F\, d\mu$ coincides with the set of integrals over A of all Pettis integrable selectors of F.

Main problems

Set-valued Pettis integration

- ✓ set-valued Pettis integral theory pretty much studied recently [Amr98, DPM05, DPM06, EAH00, HZ02, Zia97, Zia00].

The Milestone result, [EAH00, Zia97, Zia00] and [CV77, Chapter V, §4]

If X is a separable Banach space and $F:\Omega\to cwk(X)$ a multi-function TFAE:

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Main problems

- ✓ If X is a separable Banach space and $F:\Omega \to cwk(X)$: When Pettis integrability of F equivalent Pettis integrability of $j \circ F$.
- ✓ Is there a reasonable theory of set-valued Pettis integration for X non necessarily separable.

Set-valued Pettis integration and embeddings

Theorem, [CR04] and [DPM06]

Assume that X is separable and let $F:\Omega \longrightarrow cwk(X)$ be a multi-valued function. Let us consider the following statements:

(i) $j \circ F$ is Pettis integrable;

(ii) F is Pettis integrable.

Then (i) always implies (ii) and $j((P)\int_A F\, d\mu) = \int_A j\circ F\, d\mu$ for every $A \in \Sigma$. If moreover $F(\Omega)$ is h-separable (e.g. $F(\Omega) \subset ck(X)$) then (ii) implies (i).

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Theorem, [CKR07]

For a separable Banach space X the following statements are equivalent:

(i) X has the Schur property.

(ii) $(cwk(X), h)$ is separable.

(iii) For any complete probability space (Ω, Σ, μ) and any Pettis integrable multi-function $F:\Omega \to cwk(X)$ the composition $j \circ F$ is Pettis integrable.

(iv) For any Pettis integrable multi-function $F:[0,1] \to cwk(X)$ the composition $j \circ F$ is Pettis integrable .

Set-valued Pettis integration for general Banach spaces

Definition

Let X be a separable Banach space. A multi-function $F : \Omega \to cwk(X)$ is said to be *Pettis integrable* if

- $\delta^*(x^*, F)$ is integrable for each $x^* \in X^*$;
- for each $A \in \Sigma$, there is $\int_A F \, d\mu \in cwk(X)$ such that

$$\delta^*\left(x^*, \int_A F \, d\mu\right) = \int_A \delta^*(x^*, F) \, d\mu \quad \text{for every } x^* \in X^*.$$

Set-valued Pettis integration for general Banach spaces

Definition

Let X be an arbitrary Banach space. A multi-function $F:\Omega\to cwk(X)$ is said to be *Pettis integrable* if

- $\delta^*(x^*,F)$ is integrable for each $x^*\in X^*$;
- for each $A\in\Sigma$, there is $\int_A F\,d\mu\in cwk(X)$ such that

$$\delta^*\left(x^*,\int_A F\,d\mu\right)=\int_A\delta^*(x^*,F)\,d\mu\quad\text{for every } x^*\in X^*.$$

Set-valued Pettis integration for general Banach spaces

Theorem, [CKR07New]

Let X be an Banach space and $F : \Omega \to cwk(X)$ a Pettis integrable multi-function. Then:

- every scalarly measurable selector is Pettis integrable;
- F admits a scalarly measurable selector.

Furthermore, F admits a collection $\{f_\alpha\}_{\alpha<\mathrm{dens}(X^*,w^*)}$ of Pettis integrable selectors such that

$$F(\omega) = \overline{\{f_\alpha(\omega) : \ \alpha < \mathrm{dens}(X^*, w^*)\}} \quad \text{for every } \omega \in \Omega.$$

Moreover, $\int_A F \, d\mu = \overline{IS_F(A)}$ for every $A \in \Sigma$.

Consequences

Corollary, [CKR07New]

Suppose X^* is w^*-separable. Let $F:\Omega\to cwk(X)$ be a Pettis integrable multi-function. Then $\int_A F\,d\mu = IS_F(A)$ for every $A\in\Sigma$.

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Corollary, [CKR07New]

If X is reflexive, every scalarly measurable multifunction $F : \Omega \to cwk(X)$ has a scalarly measurable selector.

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If X is reflexive, every scalarly measurable multifunction $F : \Omega \to cwk(X)$ has a scalarly measurable selector.

Corollary, [CKR07New]

If X has μ-SMSP and μ-PIP (e.g. X is separable or X is reflexive) and $F : \Omega \to cwk(X)$ a multi-function TFAE:

(i) F is Pettis integrable.

(ii) The family $W_F = \{\delta^*(x^*, F) : x^* \in B_{X^*}\}$ is uniformly integrable.

(iii) The family W_F is made up of measurable functions and any scalarly measurable selector of F is Pettis integrable.

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Lemma, [CKR07New]

Let $F : \Omega \to cwk(X)$ be a multi-function such that $\delta^*(x^*, F)$ is integrable for every $x^* \in X^*$. The following statements are equivalent:

(i) F is Pettis integrable.

(ii) For each $A \in \Sigma$, the mapping

$$\varphi_A^F : X^* \to \mathbb{R}, \quad x^* \mapsto \int_A \delta^*(x^*, F)\, d\mu,$$

is $\tau(X^*, X)$-continuous.

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Let $F : \Omega \to cwk(X)$ be a Pettis integrable multi-function. Then F admits a Pettis integrable selector.

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Lemma, [CKR07New]

Let $F, G : \Omega \to cwk(X)$ be two multi-functions such that F is Pettis integrable, G is scalarly measurable and, for each $x^* \in X^*$, we have $\delta^*(x^*, G) \leq \delta^*(x^*, F)$ μ-a.e. Then G is Pettis integrable and $\int_A G\, d\mu \subset \int_A F\, d\mu$ for every $A \in \Sigma$.

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Lemma, [Val71]

Let $F : \Omega \to cwk(X)$ be a scalarly measurable multi-function. Fix $x_0^* \in X^*$ and consider the multi-function

$$G : \Omega \to cwk(X), \quad G(\omega) := \{x \in F(\omega) : \; x_0^*(x) = \delta^*(x_0^*, F(\omega))\}.$$

Then G is scalarly measurable.

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